Esercizio 937
(File scaricato da http://www.extrabyte.info)

Calcolare l'integrale:

$$\int x^2 \cos^2 3x dx \tag{1}$$

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Soluzione
Utilizziamo la nota formula:

$$\cos^2 x = \frac{1}{2} (\cos 2x + 1)$$

cioè:

$$\cos^2 3x = \frac{1}{2} (\cos 6x + 1)$$

Quindi:

$$\begin{aligned} \int x^2 \cos^2 3x dx &= \frac{1}{2} \int x^2 (\cos 6x + 1)\, dx = \frac{1}{2} \left(\int x^2 \cos 6x + \int x^2 dx \right) \\ &= \frac{1}{2} \underbrace{\int x^2 \cos 6x dx}_{F_1(x)} + \frac{1}{6} x^3 \end{aligned} \tag{2}$$

Calcoliamo a parte $F_1(x)$:

$$F_1(x) \overset{t=6x}{\longrightarrow} F_1(t) = \frac{1}{216} \int t^2 \cos t dt = \frac{1}{216} \left(\int t^2 \sin t dt - 2 \int t \sin t dt \right)$$

$$\begin{aligned} \int t^2 \sin t dt &= \int t d(-\cos t) = -t \cos t + \int \cos t dt \\ &= -t \cos t + \sin t \\ \int t \sin t dt &= \int t d(-\cos t) = -t \cos t + \int \cos t dt \\ &= -t \cos t + \sin t \end{aligned}$$

$$F_1(t) = \frac{1}{216} \left(t^2 \sin t + 2t \cos t - 2 \sin t \right)$$

Ripristinando x:

$$F_1(x) = \frac{1}{216} \left(36x^2 \sin 6x + 12x \cos 6x - 2 \sin 6x \right)$$

Sostituendo nella (2):

$$\int x^2 \cos^2 3x dx = \frac{1}{6} \left(x^3 + \frac{x^2}{2} \sin 6x + \frac{x}{6} \cos 6x - \frac{1}{36} \sin 6x \right) + C$$