Esercizio 1052
(File scaricato da http://www.extrabyte.info)

Calcolare l'integrale definito

$$\int_0^1 \frac{x}{x^2+3x+2}dx \tag{1}$$

Soluzione
Calcoliamo prima l'integrale indefinito:

$$F(x) = \int \frac{x}{x^2+3x+2}dx$$

Scriviamo:

$$x = a\frac{d}{dx}\left(x^2+3x+2\right)+b,$$

con a, b coefficienti indeterminati:

$$\begin{aligned} x &= 2ax+3a+b \\ &\Longrightarrow a = \frac{1}{2}, b = -\frac{3}{2} \end{aligned}$$

Quindi:

$$\begin{aligned} F(x) &= \frac{1}{2}\int \frac{d\left(x^2+3x+2\right)}{x^2+3x+2} - \frac{3}{2}\underbrace{\int \frac{dx}{x^2+3x+2}}_{=F_1(x)} \tag{2} \\ &= \frac{1}{2}\ln\left|x^2+3x+2\right| - \frac{3}{2}F_1(x) \end{aligned}$$

Calcoliamo

$$F_1(x) = \int \frac{dx}{x^2+3x+2}$$

A tale scopo scriviamo:

$$\begin{aligned} x^2+3x+2 &= (x+k)^2+l = x^2+2kx+k^2+l \\ &\Longrightarrow \begin{cases} 2k=3 \\ l+k^2=2 \end{cases} \Longrightarrow k=\frac{3}{2},\ l=-\frac{1}{4} \\ &\Longrightarrow x^2+3x+2 = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} = \frac{1}{4}\left[(2x+3)^2-1\right] \end{aligned}$$

L'integrale diventa:

$$\begin{aligned} F_1(x) &= 2\int \frac{d(2x+3)}{(2x+3)^2-1} = \ln\left|\frac{2x+3-1}{2x+3+1}\right| + C_1 \\ &= \ln\left|\frac{x+1}{x+2}\right| + C \end{aligned}$$

Sostituendo nella (2):

$$F(x) = \frac{1}{2}\ln\left|x^2+3x+2\right| - \frac{3}{2}\ln\left|\frac{x+1}{x+2}\right| + C$$

Osservando che $x^2+3x+2 = (x+2)(x+1)$:

$$\begin{aligned} F(x) &= \frac{1}{2}\ln|x+2| + \frac{1}{2}\ln|x+1| - \frac{3}{2}\ln|x+1| + \frac{3}{2}\ln|x+2| + C \\ &= 2\ln|x+2| - \ln|x+1| + C \end{aligned}$$

E attraverso questa espressione siamo in grado di valutare l'integrale definito:

$$\begin{aligned} \int_0^1 \frac{x}{x^2+3x+2}dx &= \left[2\ln|x+2| - \ln|x+1|\right]_0^1 \\ &= 2\ln 3 - 3\ln 2 \\ &= \ln\frac{9}{8} \end{aligned}$$