Esercizio 1035
(File scaricato da http://www.extrabyte.info)

Calcolare il seguente integrale:

$$\int \frac{5x-4}{3x^2-7x+11}dx \tag{1}$$

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Soluzione
Scriviamo:

$$\begin{aligned}
5x-4 &= a\frac{d}{dx}\left(3x^2-7x+11\right)+c \\
&= a\left(6x-7\right)+c \\
&\Longrightarrow \left\{\begin{array}{c} 6a=5 \\ c-7a=-4 \end{array}\right. \Longrightarrow a=\frac{5}{6}, \quad c=\frac{11}{6} \\
&\Longrightarrow 5x-4=\frac{5}{6}\frac{d}{dx}\left(3x^2-7x+11\right)+\frac{11}{6}
\end{aligned}$$

per cui:

$$\int \frac{5x-4}{3x^2-7x+1}dx=\frac{5}{6}\ln\left(3x^2-7x+11\right)+\frac{11}{6}F\left(x\right),$$

essendo:

$$F\left(x\right)\stackrel{def}{=}\int\frac{dx}{3x^2-7x+11}$$

Per calcolare $F\left(x\right)$ scriviamo:

$$\begin{aligned}
3x^2-7x+11 &= 3\left(x+k\right)^2+l=3x^2+6kx+3k^2+l \\
&\Longrightarrow \left\{\begin{array}{c} 6k=-7 \\ l+3k^2=11 \end{array}\right. \Longrightarrow k=-\frac{7}{6}, \, l=\frac{83}{12} \\
&\Longrightarrow 3x^2-7x+11=3\left(x-\frac{7}{6}\right)^2+\frac{83}{12} \\
&= \frac{83}{12}\left[\left(\frac{6x-7}{\sqrt{83}}\right)^2+1\right],
\end{aligned}$$

per cui:

$$\begin{aligned}
\int \frac{dx}{3x^2 - 7x + 11} &= \frac{12}{83} \int \frac{dx}{\left(\frac{6x-7}{\sqrt{83}}\right)^2 + 1} \\
&= \frac{2}{\sqrt{83}} \int \frac{d\left(\frac{6x-7}{\sqrt{83}}\right)}{1 + \left(\frac{6x-7}{\sqrt{83}}\right)^2} \\
&= \frac{2}{\sqrt{83}} \arctan\left(\frac{6x-7}{\sqrt{83}}\right) + C_1
\end{aligned}$$

Quindi:

$$\int \frac{5x-4}{3x^2 - 7x + 11} dx = \frac{5}{6} \ln\left(3x^2 - 7x + 11\right) + \frac{11}{2\sqrt{83}} \arctan\left(\frac{6x-7}{\sqrt{83}}\right) + C$$